while the values of $\nu(\pi_i)$ and $\beta(\pi_i)$ are given in the table:

i	$\nu(\pi_i)$	$\beta(\pi_i)$
1	6	6
2	2	3
3	3	2

Hence we have

$$P_3 = \frac{1}{2}\left[\frac{2^9}{6^2} + \frac{2^5}{2^2} + \frac{2^3}{3^2} + \frac{2^7}{12} + \frac{2^4}{18} + \frac{2^3}{6}\right] + \frac{1}{2}\left[\frac{2^6}{6} + \frac{2^3}{2} + \frac{2^2}{3}\right] = 26.$$

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Evaluation of the Zeros of Cross-Product Bessel Functions

By L. Jackson Laslett and William Lewish

1. Introduction. There is considerable interest in the zeros of certain cross-product Bessel functions which arise in solving Bessel's equation subject to Dirichlet or Neumann boundary conditions at $r = a, b$,

$$J_n(qa)Y_n(qb) - J_n(qb)Y_n(qa) = 0 \tag{1a}$$

or

$$J_n'(qa)Y_n'(qb) - J_n'(qb)Y_n'(qa) = 0, \tag{1b}$$

because of their well-known application in physical or engineering problems for which the use of cylindrical coordinates is appropriate. In many instances attention may be directed primarily to the zeros of such functions when n is not large because of the interest in the lower-order modes which are possible in the physical problem under consideration, but cases may also arise in which the higher-order modes will warrant attention in order to determine the circumstances in which such possibly unwanted modes may become excited.

Solutions to (1a) and (1b) have been discussed by a number of writers [1], [6], and results presented in the form of algebraic formulas, in tables, or graphically. For application to problems in which $(b - a)/(b + a)$ is small and in which n may

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be large, however, it appeared appropriate to make an independent investigation of the initial roots of (1a) and (1b) by study of characteristic solutions of Bessel's equation in the interval $a \leqq r \leqq b$ without explicit reference to the usual Bessel and Neumann functions. Approximate analytic formulas have been obtained from which estimates may be made of the characteristic values, for the case of the first Dirichlet root and for the first two roots subject to the Neumann boundary condition, and an independent numerical determination of the characteristic values and characteristic functions has been made with the CYCLONE electronic digital computer at Iowa State University for cases in which $(b - a)/(b + a)$ was given the values 0.001, 0.01, and 0.1. It is the purpose of the present note to summarize the results of this investigation, for which more detailed results will be available elsewhere (see Section 5).

2. Transformation of Bessel's Equation. It may be noted that, due to the nature of the customary Bessel functions of high order, and in particular because the function J_n remains quite small until its argument is comparable to its order, the lowest characteristic values, q, will be in the neighborhood of n/b for n large. For this reason, and to focus attention on the interval $a \leqq r \leqq b$, it is convenient to define

$$\eta = \frac{b - a}{b + a}, \tag{2a}$$

$$\delta = \eta^2 \left[\left(q \frac{b + a}{2} \right)^2 - n^2 \right], \tag{2b}$$

and

$$x = 2 \frac{r - (b + a)/2}{b - a}. \tag{2c}$$

In terms of these quantities,

$$r = \frac{b + a}{2} (1 + \eta x), \quad \text{with} \quad -1 \leqq x \leqq 1, \tag{3}$$

and Bessel's equation assumes the form

$$\frac{d}{dx}\left[(1 + \eta x) \frac{dZ}{dx} \right] + \left[\delta(1 + \eta x) + \frac{2 + \eta x}{1 + \eta x} \cdot \eta^3 n^2 \cdot x \right] Z = 0. \tag{4}$$

The solutions to (4) which are of interest are those for which the Dirichlet boundary condition ($Z = 0$) or, alternatively, the Neumann boundary condition ($dZ/dx = 0$) applies at $x = \pm 1$. When the Dirichlet boundary condition is applied, it may be convenient for some purposes to make the transformation

$$S = (1 + \eta x)^{1/2} Z, \tag{5}$$

in terms of which (4) may be written

$$\frac{d^2S}{dx^2} + \left[\delta + \frac{(\eta^2/4) + \eta^3 n^2 (2 + \eta x) x}{(1 + \eta x)^2} \right] S = 0, \tag{6}$$

with $S(\pm 1) = 0$.

Physically, it is seen that the quantity η which is introduced here denotes the ratio of the width $(b - a)$ to the mean diameter $(b + a)$ of an annular region. For η only slightly less than unity, the annular region extends substantially from $r = 0$ to $r = b$ and the roots $q \frac{b + a}{2}$ of (1a) or (1b) may then be expected to become one-half the corresponding roots, μ, of the simpler equations $J_n(\mu) = 0$ or $J_n'(\mu) = 0$, respectively.

For $\eta \ll 1$, the terms in (4) or (6) which contain η, save in some cases those which involve the combination $\eta^3 n^2$, may either be ignored in determining simple analytic formulas for δ or may be treated as a perturbation.

3. Approximate Analytic Formulas. For $\eta \ll 1$, the characteristic values, δ, for (4) or (6) may be obtained by a perturbation method [7] in which the unperturbed equation is taken as simple harmonic, provided n is not too large. In this way we find

(7a) For the first Neumann root: $\delta \sim \frac{1}{3} \eta^4 n^2 - \frac{8}{15} \eta^6 n^4,$

(7b) For the first Dirichlet root: $\delta \sim \left(\frac{\pi}{2}\right)^2 - \frac{\eta^2}{4} + \left(1 - \frac{6}{\pi^2}\right)\left(n^2 - \frac{1}{4}\right)\eta^4,$

(7c) For the second Neumann root: $\delta \sim \left(\frac{\pi}{2}\right)^2 + \frac{3}{4}\eta^2 + \left(1 + \frac{10}{\pi^2}\right)\eta^4 n^2.$

The nature of the characteristic solution associated with the first Neumann root is such that it is very nearly constant when $\eta^3 n^2$ is small. In such cases the form of the solution is approximately given by $Z \sim 1 + \eta^3 n^2 \left(x - \frac{x^3}{3}\right)$. Similarly, the first Dirichlet and second Neumann solutions are respectively of the general character $\cos \frac{\pi}{2} x$ or $\sin \frac{\pi}{2} x$. The region of applicability of (7a–c) may be considered to be that for which $\eta^3 n^2 \ll 1$; of equal or greater interest, however, are the results for the case $\eta^3 n^2 > 1$, which is discussed below.

In cases for which $\eta^3 n^2$ is not small, but $\eta \ll 1$, it may suffice to replace (4) by

$$\frac{d^2 Z}{dx^2} + [\delta + 2\eta^3 n^2 \cdot x] Z = 0. \tag{8}$$

Solutions of this approximate equation may be written explicitly in terms of Bessel and Neumann functions of order $\frac{1}{3}$. It then follows, moreover, that for $\eta^3 n^2$ at least somewhat greater than unity (*e.g.*, $\eta^3 n^2 > 6$) the solution of interest is substantially

$$Z \sim \begin{cases} \xi^{1/2} \left[J_{1/3}\left(\frac{\xi^{3/2}}{3\eta^3 n^2}\right) + J_{-1/3}\left(\frac{\xi^{3/2}}{3\eta^3 n^2}\right) \right], & \text{for } \xi \geqq 0, \\ \frac{3^{1/2}}{2} i^{4/3} \mid \xi \mid^{1/2} H_{1/3}^{(1)}\left(i \frac{\mid \xi \mid^{3/2}}{3\eta^3 n^2}\right), & \text{for } \xi \leqq 0, \end{cases} \tag{9}$$

where ξ denotes $\delta + 2\eta^3 n^2 x$, since the first Hankel function then becomes sufficiently small at $x = -1$ as to satisfy adequately the boundary condition normally imposed at that point. The characteristic values, δ, may then be estimated by application of the desired boundary condition at $x = 1$, aided by tables of $J_{\pm 1/3}$ and $J_{\pm 2/3}$ [8], [9],

(10a) For the first Neumann root: $\delta \sim -2\eta^3 n^2 + 1.61724\eta^2 n^{4/3}$,

(10b) For the first Dirichlet root: $\delta \sim -2\eta^3 n^2 + 3.71151\eta^2 n^{4/3}$,

(10c) For the second Neumann root: $\delta \sim -2\eta^3 n^2 + 5.15619\eta^2 n^{4/3}$.

The numerical constants which appear in (10a, b) are seen to be, as expected, twice the numerical coefficients given in series developments for the first maximum and first zero of J_n when n is large [9 (Sect. 15.83, p. 521)]; [10 (Sect. VIII.3.6, p. 143)]. Characteristic values for solutions to (8) must necessarily be somewhat less negative than $-2\eta^3 n^2$ in order that the coefficient of Z be positive for *some* values of x in the interval $-1 \leqq x \leqq 1$. For $\eta^3 n^2$ large, the characteristic solutions are relatively large only for values of x near unity, in a region whose width is roughly two or three times $(\eta^3 n^2)^{-1/3}$.

4. Computational Results. The differential equation (4), suitably scaled, was integrated with the CYCLONE digital computer at Iowa State University, using the Runge-Kutta process [11], [12]. Runs were made for several values of n, with η given in turn the values 0.001, 0.01, and 0.1. In each case the value of δ was adjusted, by trial, to give solutions satisfying the desired Dirichlet or Neumann boundary conditions. A larger number of integration steps was employed to traverse the interval $-1 \leqq x \leqq 1$ in cases in which $\eta^3 n^2$ was large, since more rapid changes of the function occur in certain portions of that interval in such cases. The effect of truncation error was found, by tests in which the interval size was halved, to be sufficiently small that use of the finer interval only affected the final value for the function or its derivative (in the Dirichlet or Neumann cases, respectively) by less than 10^{-6} of the maximum value and the consequent error in δ could thus be judged when tabulating the results of the investigation.

The characteristic values δ determined computationally are listed in Table I. By comparing calculated values of δ obtained for (7a–c) and (10a–c) with the values in Table 1, the accuracy of (7a–c) and (10a–c) can be ascertained. See Table VI [13]. Figure 1 depicts the nature of the associated characteristic functions, for $n = 0.01$, for various representative values of n. Since the contribution from δ makes a relatively small change in the characteristic value for the original Bessel equation when n is large, use of (2b) in connection with the values of δ given in Table I should afford accurate characteristic values for q in such cases. In the application to physical problems it is interesting to note from Figure 1 the features mentioned in Section 3, namely that at small n the first Neumann solution does not show a pronounced variation with x and the other characteristic solutions have approximately the form of circular functions, while at large n the characteristic solutions become large only in a small interval near $x = 1$.

5. Availability of Detailed Results. The analytic work of Section 3 is presented in greater detail, and the computational results reproduced directly from the teleprinter output of the CYCLONE, in an Ames Laboratory report [13]. The report

TABLE 1

Values of δ for the first Neumann eigenvalue (N_1), the first Dirichlet eigenvalue (D_1), and the second Neumann eigenvalue (N_2).

η Root	$\eta = 0.001$ N_1	D_1	N_2	$\eta = 0.01$ N_1	D_1	N_2	$\eta = 0.1$ N_1	D_1	N_2	$\eta = 1.0$* N_1	D_1	N_2
$n = 0$	0	2.4674011	2.46740209	0	2.467376	2.467476	0	2.4648915	2.4749309	(0)	1.445797	3.670493
½	0	2.4674011	2.46740209	0	2.467376	2.467476	0.00000839	2.4649013	2.474989	0.089633	2.2174011	5.049703
1	0	2.4674011	2.46740209	0	2.467376	2.467476	0.0000330	2.4649308	2.4751343	−0.152511	2.670493	6.106071
2	0	2.4674011	2.46740209	0	2.467376	2.467476	0.0001255	2.4650481	2.4757499	−1.667909	2.593654	7.243048
5	0	2.4674011	2.46740209	0	2.467376	2.467476	0.0005019	2.4658332	2.4802714	−14.709967	−5.765268	2.666867
10	0	2.4674011	2.46740209	0	2.467377	2.467478	−0.0019893	2.4681254	2.4994189	−65.361612	−47.614970	−32.367035
20	0	2.4674021	2.46740209	.0000012	2.467378	2.467484	−0.0680405	2.4693426	2.6189112	−276.5774	−238.492	−208.010
30							−0.335704	2.4435640	2.9276509			
40							−0.905569	2.3513268	3.4493650			
50	0	2.4674011	2.46740210	.0000050	2.467386	2.467529	−1.799205	2.143523	4.0878274			
75							−5.418144	0.8024951	5.142091			
100	0	2.4674011	2.46740211	−0.000020	2.467409	2.467720	−11.009312	−2.103247	4.361139			
150							−28.210824	−13.285380	−2.541564			
200	0.00000002	2.4674011	2.46740217	−0.000720	2.467421	2.468957	−53.56333	−31.956122	−16.514506			
500	0.00000004	2.4674015	2.4674026	−0.032070	2.463969	2.498479						
1000	−0.00000020	2.4674014	2.4674046	−0.456025	2.401502	2.835214						
1500				−1.717081	2.130264	3.657827						
2000	−0.00000720	2.4674015	2.4674169	−3.876713	1.441630	4.535002						
2500				−6.921675	0.145494	4.843058						
3000				−10.857817	−1.875544	4.303889						
4000				−21.440133	−8.287472	0.828077						
5000	−0.00032495	2.467367	2.4677163	−35.682279	−17.993822	−5.740178						
10000	−0.005290	2.466738	2.4718184									
20000	−0.082838	2.456331	2.533433									
50000	−2.030430	2.042788	3.787091									
75000	−6.127374	0.515508	4.811675									
80110		0										
100000	−12.476797	−2.763631	3.907135									
130953			0									
150000	−32.066830	−15.395587	−3.890509									
200000	−60.999918	−36.537911	−19.657327									

* From published tables [8].

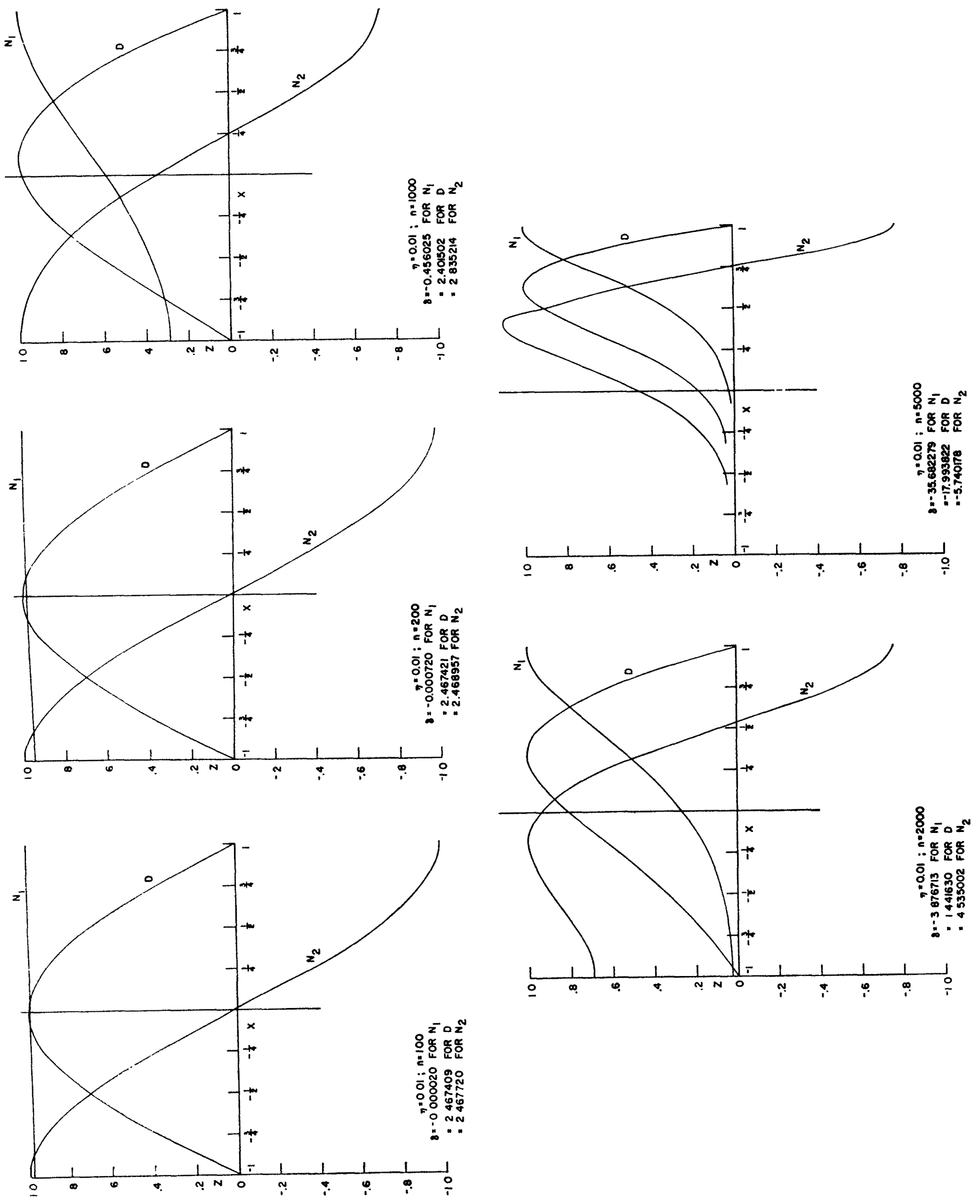

FIG. 1–Characteristic functions for $\eta = .01$ illustrating the effect of n for the first Neumann eigenvalue (N_1), the first Dirichlet eigenvalue (D), and the second Neumann eigenvalue (N_2).

also includes approximate values of $\int_{-1}^{1} Z^2 dx$, suitably normalized with respect to the value of Z or dZ/dx at $x = 0$ and at $x = 1$, for $\eta = 0.0001$ and for representative values of $\eta^3 n^2$ in the range 0 through 20. This report is available from the Office of Technical Services, U. S. Department of Commerce. Two copies of the report have been deposited in the file of Unpublished Mathematical Tables which is maintained by *Mathematics of Computation* and may be made available on loan to interested individuals.

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On the Computation of Lommel's Functions of Two Variables

By J. Boersma

In 1942 Zernike and Nijboer [1], [2] introduced a new expansion of Lommel's functions of two variables in connection with calculating the diffraction integral of a circular aperture. In this article it is shown that this expansion is very well suited for the computation of these functions. (The author is much indebted to Dr. Bottema of the Physical Laboratory of the University of Groningen, who drew his attention to this formula.)

Lommel's functions of two variables are defined in the following way (Cf. [3],

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